Tentamen Signal Analysis, 31/1/06, room 5116.0116, 9.00-12.15

Please write your name and student number on each sheet; answer either in Dutch or English.

Question 1

a. The function $f(t)$ is defined by $f(t) = e^{-t/\tau}$ for $t \geq 0$ and $f(t) = 0$ for $t < 0$, and the function $g(t)$ is defined by $g(t) = e^{t/\tau}$ for $t \leq 0$ and $g(t) = 0$ for $t > 0$. Calculate the Fourier transforms of $f(t)$ and $g(t)$.

b. Prove that for any function $\varphi(t)$ the relationship $\mathcal{F}\{\varphi(-t)\} = [\mathcal{F}\{\varphi^*(t)\}]^*$ holds, where $\mathcal{F}$ denotes the Fourier transform, and * the complex conjugate. Show that your result at **a** is consistent with this relationship.

c. With $f * g$ denoting the convolution of *f* and *g*, show that for the specific functions *f* and *g* defined in **a** $f * g = \frac{\tau}{2}(f + g)$, by first showing that $\mathcal{F}\{f\}\mathcal{F}\{g\} = \frac{\tau}{2}[\mathcal{F}\{f\} + \mathcal{F}\{g\}]$.

d. Show that $f * g = \frac{\tau}{2}(f + g)$ by evaluating $f * g$ directly in the time domain, using the definition of convolution.

Question 2

A signal *g(t)* is defined as the product of three cosines: $g(t) = \cos(2\pi f_1 t)\cos(2\pi f_2 t)\cos(2\pi f_3 t)$, with f_1=100 Hz, f_2=200 Hz, and f_3=250 Hz.

a. Which frequency components are present in *g(t)*? Sketch the positions and amplitudes of all frequency components on the frequency axis ($-\infty < f < \infty$).

Hint: you may want to use $\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$).

b. Determine the minimal sampling frequency needed to be able to fully reconstruct *g(t)* from the sampled values. Explain, using clear diagrams in the frequency domain, why this sampling frequency is sufficient.

c. Suppose you sample *g(t)* at a sampling frequency f_s=650 samples/s, and you subsequently use the sampled values to make a reconstruction $g_r(t)$ of *g(t)*. Which frequencies are present in $g_r(t)$? Explain your answer with a diagram in the frequency domain.

Question 3

a. *X* is a random variable with probability density function (pdf) $p_X(x)$. Give the cumulative distribution function (cdf) of *X*, denoted by $F_X(x)$, in terms of $p_X(x)$. Also give $p_X(x)$ in terms of $F_X(x)$.

b. A random variable *Y* is defined by a monotonically increasing function *f* of a random variable *X*, i.e., $y = f(x)$. Derive that $p_Y(y) = p_X(x)\frac{dx}{dy}$. (Remark: this is a special case of $p_Y(y) = p_X(x)\left|\frac{dx}{dy}\right|$).

c. *X* is a random variable with pdf $p_X(x) = \frac{2}{\pi}$ for $0 \leq x \leq \frac{\pi}{2}$ and $p_X(x) = 0$ elsewhere. *Y* is a random variable given by $y = \sin(x)$, and *Z* a random variable given by $z = \cos(x)$. Calculate and sketch $p_Y(y)$ and $p_Z(z)$.

d. Sketch the joint probability density function of *Y* and *Z*, $p_{Y,Z}(y,z)$. Use the definition of independence to argue that *Y* and *Z* are not independent random variables.

NB See other side for Question 4

Question 4

The random signal $s(t)$ is the sum of two statistically independent, stationary random signals $x(t)$ and $y(t)$, where $x(t)$ has an autocorrelation function $R_x(\tau) = \exp(-|\tau|)$, and $y(t)$ is zero mean white noise with a power spectral density $\mathcal{P}_y(f) = 0.1$. This signal $s(t) = x(t) + y(t)$ is passed through a linear filter with impulse response $h(t)$

$$h(t) = \frac{1}{T} \qquad \text{for } 0 \le t \le T$$

$$h(t) = 0 \qquad \text{for } t < 0 \text{ and } t > T$$

a. Calculate the power spectral density of signal $x(t)$.

b. Calculate the autocorrelation function and power spectral density of $s(t)$.

c. Find the power spectral density of the signal at the output of the linear filter.

d. Calculate the Wiener filter for optimally retrieving $x(t)$ from $s(t)$. If you were forced to use $h(t)$ as an approximation to this Wiener filter, which value would you (approximately) choose for T?